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Abstract.

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DIRAC OPERATORS AND MANIFOLDS WITH BOUNDARY *

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Key words: Calderón projector, Clifford modules, Dirac operator, elliptic boundary problems, eta-invariant, index theory, partitioned manifolds, pseudo-differential Grassmannian, spectral flow, surgery

1. Begin With Clifford Modules

There are many different concepts of a Dirac operator in global analysis: classical and twisted Dirac operators on spin manifolds; operators of Dirac type with a square with scalar principal symbol; generalized (or compatible) Dirac operators defined by arbitrary (or compatible) connections on bundles of Clifford modules over Riemannian manifolds; full and split (odd-parity) Dirac operators; boundary Dirac operators; etc. The concepts depend on various geometrical features like dimension parity, orientation and chirality, almost complex structure, and suitable boundary. Each definition has its own merits and range of application.

Let X be a compact smooth oriented manifold (with or without boundary) with Riemannian metric g. Let $\dim X = n$. Let S be a complex vector bundle over X of Clifford modules; i.e. we have a representation

$$\mathbf{c} : \mathrm{C}\ell(X) \longrightarrow \mathrm{Hom}(S, S)$$

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with

$$c(v)^2 = -\|v\|^2 \mathrm{Id}_{S_x} \qquad \text{for } v \in TX_x \text{ and } x \in X. \tag{1}$$

Recall that the Clifford bundle $C\ell(X)$ consists of the Clifford algebras $C\ell(TX_x, g_x)$, $x \in X$, which are associative algebras with unit generated by TX_x and subject to the relation $v \cdot w + w \cdot v = -2g_x(v,w)$. We shall call c *left Clifford multiplication* and occasionally write

$$c : C^\infty(X; TX \otimes S) \longrightarrow C^\infty(X; S).$$

We may assume that S is equipped with a Hermitian metric which makes Clifford multiplication skew-adjoint, i.e. $c(v)^* = -c(v)$ for all $v \in TX_x$.

Definition 1 A connection $D : C^\infty(X;S) \longrightarrow C^\infty(X; T^*X \otimes S)$ for S will be called *compatible* with the Clifford module structure of S, if it is *Leibnizian*, i.e. it satisfies the product rule

$$\partial_v \langle s, s' \rangle = \langle D_v s, s' \rangle + \langle s, D_v s' \rangle, \tag{2}$$

and if $Dc = 0$, i.e D is a *module derivation* with

$$(Dc)(v)(s) = D(c(v)s) - c(D^g v)s - c(v)(Ds) = 0, \tag{3}$$

where D^g denotes the Levi-Civita connection on X.

Patching locally constructed spin connections together proves

Theorem 2 (Branson, Gilkey [12]) *There exist compatible connections on S which extend the Riemannian connection on X to S.*

Definition 3 Let $A : C^\infty(X;S) \longrightarrow C^\infty(X;S)$ be a linear differential operator of first order operating on smooth sections of a $C\ell(X)$-module S.
(a) We call A an *operator of Dirac type*, if the principal symbol of its square is defining the Riemannian metric:

$$\sigma_{A^2}(x,\xi) = \sum_{\mu,\nu=1}^{n} g^{\mu\nu}(x)\xi_\mu\xi_\nu. \tag{4}$$

(b) We call A a *generalized Dirac operator*, if it can be written as $A = c \circ J \circ D$, where D is a (not necessarily compatible) connection and

$$J : C^\infty(X; T^*X \otimes S) \cong C^\infty(X; TX \otimes S)$$

denotes the canonical identification. In terms of a local orthonormal frame $v_1, \ldots, v_n$ of TX we then have

$$As|_x = \sum_{\nu=1}^{n} c(v_\nu)(D_{v_\nu} s)|_x.$$

(c) We call A a *(compatible) Dirac operator*, if it can be written as $A = c \circ J \circ D$, where D is a compatible connection.

Note. In this article we deal with compatible Dirac operators. However, most of the arguments remain valid for generalized non-compatible Dirac operators like the Dolbeault complex or, even more general, operators of Dirac type.

Clearly all (total) Dirac operators are elliptic and formally self-adjoint with a *Green's formula*

$$(As, s') - (s, As') = -\int_Y G(y)\langle s|_Y, s'|_Y\rangle, \tag{5}$$

where $G(y) := c(\mathbf{n})$ denotes Clifford multiplication by the inward unit tangent vector.

For even n the splitting $C\ell(X) = C\ell^+(X) \oplus C\ell^-(X)$ of the Clifford bundles induces a corresponding splitting of $S = S^+ \oplus S^-$ and a *chiral decomposition*

$$A = \begin{pmatrix} 0 & A^- \\ A^+ & 0 \end{pmatrix}.$$

The *partial Dirac operators* $A^\pm$ are especially interesting in index theory since they are also elliptic, but in general not self-adjoint and provide interesting integer-valued invariants as their indices. Like the Cauchy-Riemann operator $\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y) = \frac{1}{2}e^{i\varphi}(\partial_r + \frac{i}{r}\partial_\varphi)$ on the punctured two-disc all partial Dirac operators A^+ can be written in product form

$$A^+ = G(u, y)(\partial_u + B_u) \tag{6}$$

close to the boundary, where u denotes the inward oriented normal coordinate. Notice that the Clifford multiplication $G(u, y)$ defines a unitary morphism $S^+|_Y \to S^-|_Y$ and that $\{B_u\}$ is a family of self-adjoint (total) Dirac operators over Y. In the cylindrical case of a product metric close to Y the operators B_u and the morphisms $G(u, y)$ are independent of u.

2. Three Analysis Tools

We shall build our analysis on three basic properties which are not widely known and seem partly overlooked (2.1 and 2.3), partly insufficiently exploited (2.2) in the literature on partial differential equations.

2.1. Invertible Extension

Clifford multiplication by the inward normal vector gives a natural clutching of S^+ over one copy of X with S^- over a second copy of X to a smooth bundle $\widetilde{S^+}$ over the closed double $\widetilde{X}$. As observed in [39], the product forms of A^+ and A^- fit together over the boundary and provide a Dirac operator

$$\widetilde{A^+} := A^+ \cup A^- : C^\infty(\widetilde{X}; \widetilde{S^+}) \longrightarrow C^\infty(\widetilde{X}; \widetilde{S^-}).$$

Clearly $(A^+ \cup A^-)^* = A^- \cup A^+$; hence index $\widetilde{A^+} = 0$. It turns out that $\widetilde{A^+}$ is invertible with a pseudo-differential elliptic inverse $(\widetilde{A^+})^{-1}$. Of course A^+ is not invertible and $r^+(\widetilde{A^+})^{-1}e^+A^+ \neq \mathrm{Id}$, where $e^+ : L^2(X; S^+) \to L^2(\widetilde{X}; \widetilde{S^+})$ denotes the extension by zero operator and $r^+ : \mathcal{H}^t(\widetilde{X}; \widetilde{S^+}) \to \mathcal{H}^t(X; S^+)$ the natural restriction operator for Sobolev spaces, t real.

2.2. Calderón Projector

The next piece is the Calderón projector. It is a pseudo-differential projection onto the Cauchy data spaces (announced in Calderón [14] and proved in Seeley [34] in great generality). We define the *Cauchy data spaces*

$$H_+(A^+) := \{s|_Y \mid s \in C^\infty(X; S^+) \text{ and } A^+ s = 0 \text{ in } X \setminus Y\}$$

and, for real t,

$$H_+(A^+, t) := \text{ closure of } H_+(A^+) \text{ in } \mathcal{H}^{t-\frac{1}{2}}(S^+|_Y);$$

and the *null spaces*

$$\ker_+(A^+, t) := \{s \in \mathcal{H}^t(X; S^+) \mid A^+ s = 0 \text{ in } X \setminus Y\}.$$

The null spaces consist of sections which are distributional for negative t; which by elliptic regularity are smooth in the interior; and which by a Riesz operator argument can be shown to possess a trace $\gamma_0(s)$ over the boundary in $\mathcal{H}^{t-\frac{1}{2}}(Y; S^+|_Y)$.

First we construct a *Poisson type operator*

$$K_+ := r^+(\widetilde{A^+})^{-1}(\gamma_0^-)^* G : C^\infty(S^+|_Y) \longrightarrow C^\infty(S^+|_{X \setminus Y}).$$

It extends to a continuous mapping $\mathcal{H}^{t-\frac{1}{2}}(S^+|_Y) \to \ker_+(A^+, t)$ which is a bijection, if restricted to $H_+(A^+, t)$. Then we obtain the *Calderón projector* by taking the traces

$$\mathcal{P}_+ := \lim_{u \to 0_+} \gamma_u K_+$$

and similarly $\mathcal{P}_- := -\lim_{u \to 0_-} \gamma_u K_+$. Then $\mathcal{P}_+$ is a pseudo-differential projection with $\text{range}(\mathcal{P}_+) = H_+(A^+)$ and $\mathcal{P}_+ + \mathcal{P}_- = \text{Id}$.

The principal symbol p_+ of the Calderón projector is the projection onto the eigenspaces of the principal symbol $b(y, \zeta)$ of B_0 corresponding to non-negative eigenvalues. Hence it coincides with the principal symbol of the spectral projection $\text{P}_{\geq}(B_0)$. We call the space of pseudo-differential projections with the same principal symbol p_+ the *Grassmannian* Gr_{p_+}. It has enumerable many connected components; two projections P_1, P_2 belong to the same component, if and only if the *virtual codimension*

$$\text{i}(P_2, P_1) := \text{index}\{P_2 P_1 : \text{range} P_1 \to \text{range} P_2\} \tag{7}$$

of P_2 in P_1 vanishes; the higher homotopy groups of each connected component are given by Bott periodicity.

2.3. Twisted Orthogonality of Cauchy Data Spaces

It is a nice feature of the Clifford multiplication G, first observed in [9], that it describes the orthogonal complement of the Cauchy data space of A^+ by

$$G^{-1}(H_+(A^-)) = (H_+(A^+))^\perp \tag{8}$$

and provides a short exact sequence

$$0 \to G^{-1}(H_+(A^-, t)) \hookrightarrow \mathcal{H}^t(Y; S^+|_Y) \xrightarrow{K_+} \ker_+(A^+, t) \to 0.$$

3. Some Basic Results

To get a closed operator, full regularity of the solutions, and a finite integer-valued index for Dirac operators over manifolds with boundary, we must impose global elliptic boundary conditions.

Definition 4 A pseudo-differential operator $R : C^\infty(Y; S^+|_Y) \to C^\infty(Y; V)$ of order 0 defines an *elliptic boundary condition* for A^+, if range$R^{(t)}$ is closed in $\mathcal{H}^t(Y;V)$ and range$(r) = $ range$(rp_+) = $ range(p_+). Here $R^{(t)}$ denotes the continuous extension of R to the t-th Sobolov space and r the principal symbol of R.

Examples 5 Typical examples are the *Atiyah-Patodi-Singer boundary condition* defined by the spectral projection $P_\geq$ of the boundary Dirac operator B_0; *generalized Atiyah-Patodi-Singer boundary conditions* defined by projections belonging to the Grassmannian Gr_{p_+}; and *local elliptic boundary conditions* characterized by the additional condition that the range of r can be written as the lifting of the vector bundle V under the natural projection $T^*Y \setminus 0 \to Y$. For even-dimensional X the Clifford multiplication becomes non-trivial and excludes the existence of local elliptic boundary conditions for A and A^+ (though not for systems); for odd-dimensional X we have natural local elliptic boundary conditions $\Pi_\pm$ defined by the orthogonal projection of $S|_Y$ onto $(S|_Y)^\pm$.

Theorem 6 *The operator* $A_R^+ : \mathrm{dom} A_R^+ \to L^2(X; S^-)$, *which acts like* A^+ *and is determined by*

$$\mathrm{dom}\, A_R^+ := \{s \in \mathcal{H}^1(X; S^+) \mid R^{(0)}(\gamma_0 s) = 0\},$$

is a Fredholm operator from $L^2(X; S^+)$ *to* $L^2(X; S^-)$ *with* $\mathrm{index}\, A_R^+ = \mathrm{i}(\mathcal{R}, \mathcal{P}_+)$.

Proof A first proof was sketched in Seeley [35]. We show how easy it is in our context: The subspace $\ker A_R^+$ of $\mathcal{H}^1(X; S^+)$ consists of smooth sections, since $A^+ s = 0$ and $R(\gamma_0 s) = 0$ imply that $h := \gamma_0 s$ belongs to the kernel of the boundary integral $\mathcal{R}\mathcal{P}_+ : H_+(A^+, 1) \to \mathrm{range}\mathcal{R}^{(1)}$ which is contained in the kernel of the 'fan' $(\mathrm{Id} - \mathcal{P}_+) + \mathcal{P}_+ \mathcal{R}^* \mathcal{R} \mathcal{P}_+$; the fan is elliptic by the symbol compatibility condition of Definition 4; hence h is smooth; hence also $s = K_+ h$ is smooth. (The concept of an *elliptic fan* is due to M. Birman and A. Solomyak [6]). The argument establishes the isomorphism

$$\ker A_R^+ \cong \ker\{\mathcal{R}\mathcal{P}_+ : H_+ \longrightarrow \mathrm{range}(\mathcal{R})\}$$

and the finite dimension of the kernel. That A_R^+ is a closed L^2 realization can be deduced from the explicit description of a left parametrix for A^+ by

$$(r^+ (\widetilde{A^+})^{-1} e^+) A^+ = \mathrm{Id} - K_+ \gamma_0,$$

which is a direct consequence of the Calderón construction. Now all is simple: Since we have an explicit description of the adjoint operator

$$(A_R^+)^* = A^-_{G(\mathrm{Id}-R)G^*}$$

(if R is a projection; otherwise replace R by the orthogonal projection onto the range of R^*) and since the range of A_R^+ is closed in L^2, we also have an explicit description of the cokernel of A_R^+ which, by the Clifford rotation of Cauchy data spaces of (8), can be identified with the cokernel of the boundary integral. □

It follows from the topology of Gr_{p_+} that for generalized Atiyah-Patodi-Singer boundary conditions index A_R^+ vanishes, if and only if the projection R belongs to the same connected component of Gr_{p_+} as the Calderón projector $\mathcal{P}_+$. More generally, we obtain two explicit versions of the classical *Agronovič-Dynin formula* (see [1]).

Theorem 7 *For two projections R_1 and $R_2 \in \mathrm{Gr}_{p_+}$ we have*

$$\mathrm{index}(A_{R_1}^+) - \mathrm{index}(A_{R_2}^+) = \mathbf{i}(R_1, R_2); \tag{9}$$

and for two local elliptic boundary conditions $R_j : C^\infty(Y; S^+|_Y) \to C^\infty(Y; V_j)$, $j = 1, 2$, we have

$$\mathrm{index}(A_{R_1}^+) - \mathrm{index}(A_{R_2}^+) = \mathrm{index}\{R_1 \mathcal{P}_+ R_2 : C^\infty(Y; V_2) \to C^\infty(Y; V_1)\}. \tag{10}$$

From (9) we get a generalization of the *Atiyah-Patodi-Singer index formula* of [3]:

Theorem 8 *For $R \in \mathrm{Gr}_{p_+}$ we have*

$$\mathrm{index}\, A_R^+ = \int_X \alpha(x) - \frac{1}{2}(\eta_B(0) + \dim\ker B) + \mathbf{i}(R, \mathrm{P}_{\geq}).$$

Here $\alpha(x)$ denotes the locally defined *index density* of A^+ and

$$\eta_B(z) := \sum_{\lambda \in \mathrm{spec} B \setminus \{0\}} \mathrm{sign}\lambda |\lambda|^{-z} = \frac{1}{\Gamma(\frac{z+1}{2})} \int_0^\infty t^{\frac{z-1}{2}} \operatorname{tr}(Be^{-tB^2})\, dt \tag{11}$$

denotes the *η-function* of B. It is (i) well defined through absolute convergence for $\Re(z)$ large; (ii) it extends to a meromorphic function in the complex plane with isolated simple poles; (iii) its residues are given by a local formula; and (iv) it has a finite value at $z = 0$ (see e.g. Gilkey [16]).

Theorem 8 separates the contributions to the index from the whole manifold, from the structure on the boundary, and from the boundary condition in relation to the structure on the boundary. A special feature is that the correction term $\mathbf{i}(R, \mathrm{P}_{\geq})$ is not a homotopy invariant. It can change, e.g. under smooth deformations of the Riemannian metric, and is therefore a good candidate for more refined geometrical invariants.

To look at the geometrical aspects more closely, it is, as usual when working with Clifford algebras, appropriate to distinguish between the even- and the odd-dimensional case; and to alternate the focus between the index, the spectral flow, and the η-invariant.

4. Even Dimension. Index Theory

Let $M = X_1 \cup X_2$ be an even-dimensional closed partitioned manifold with $\partial X_1 = \partial X_2 = X_1 \cap X_2 = Y$. To illustrate the twisting of Cauchy data spaces by Clifford multiplication we prove

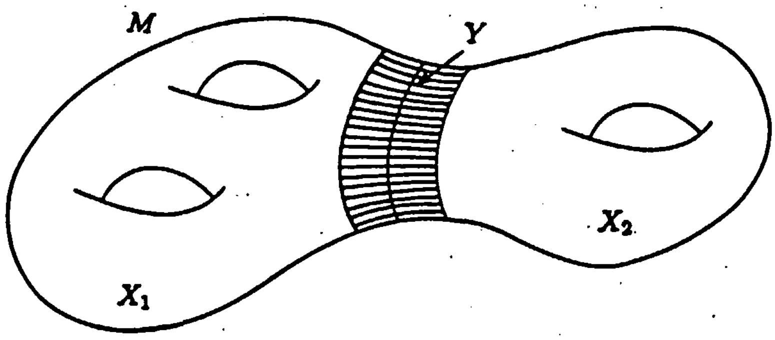

Fig. 1. Partitioned manifold

Theorem 9 (Bojarski's conjecture [7]) *Let A be a Dirac operator over M, let A^j denote its restriction to X_j, and let $\mathcal{P}(A^j)$ and $H(A^j)$ denote the corresponding Calderón projectors and L^2 closures of the Cauchy data spaces, $j = 1, 2$. Then*

$$\operatorname{index} A = \mathrm{i}(\mathrm{Id} - \mathcal{P}(A^2), \mathcal{P}(A^1)) = \operatorname{index}(H(A^1), H(A^2)).$$

Note. Notice that $\mathrm{Id} - \mathcal{P}(A^2) = G\mathcal{P}(A^1)G^*$ by (8) and recall that

$$\mathrm{i}(\mathrm{Id} - \mathcal{P}(A^2), \mathcal{P}(A^1)) := \operatorname{index}\{(\mathrm{Id} - \mathcal{P}(A^2))\mathcal{P}(A^1) : H(A^1) \longrightarrow H(A^2)^{\perp}\}$$

and

$$\operatorname{index}(H(A^1), H(A^2)) := \dim(H(A^1) \cap H(A^2)) - \dim(L^2(Y; S|_Y)/(H(A^1) + H(A^2)).$$

Note. Closed subspaces for which the two dimensions in the preceding definition are finite are called *Fredholm pairs* of subspaces.

Proof It follows from the unique continuation property for Dirac operators and Green's formula (5) that $\ker A \cong H(A^1) \cap H(A^2)$. Now we apply the Clifford rotation formula for Cauchy data spaces (8) to get $\operatorname{coker} A \cong \ker A^* \cong G(H(A^1)^{\perp}) \cap G(H(A^2)^{\perp})$. The last space is isomorphic to $H(A^1)^{\perp} \cap H(A^2)^{\perp}$ which is the orthogonal complement of $H(A^1) + H(A^2)$ in $L^2(Y; S|_Y)$. □

Replacing the Clifford multiplication with an arbitrary unitary automorphism Φ of $S|_Y$, which is *consistent* (i.e. Φ commutes with p_+), leads us to the *general linear conjugation problem*, a generalization of the classical Riemann-Hilbert problem: We are looking for couples (s_1, s_2) with

$$A^j s_j = 0 \text{ in } X_j \setminus Y \quad \text{for } j = 1, 2 \qquad \text{and} \qquad s_2|_Y = \Phi(s_1|_Y). \tag{12}$$

Let $\operatorname{index}_{\mathrm{LCP}}(A, \Phi)$ denote the difference between the dimensions of the solution spaces of the original problem (12) and of the corresponding adjoint problem. We obtain

Theorem 10 ([8], [9])

$$\operatorname{index}_{\mathrm{LCP}}(A, \Phi) = \operatorname{index}(\mathrm{Id} - \mathcal{P}(A^2))\Phi\mathcal{P}(A^1) = \operatorname{index} A + \operatorname{index} \mathcal{P}(A^1)\Phi\mathcal{P}(A^1).$$

Instead of the index of the generalized Toeplitz operator $\mathcal{P}(A^1)\Phi\mathcal{P}(A^1)$ we can calculate the index of the elliptic pseudo-differential operator $(\mathrm{Id}-\mathcal{P}(A^1))-\Phi\mathcal{P}(A^1)$ of order 0 over Y applying the Atiyah-Singer index theorem. Or we determine the spectral flow $\mathrm{sf}\{B_u\}$ of any smooth family of elliptic self-adjoint operators connecting the Dirac operator B_0 with its gauge transform $\Phi^{-1}B_0\Phi$. Recall that the *spectral flow* is the difference between the number of eigenvalues, which change the sign from $-$ to $+$ as u goes from 0 to 1, and the number of eigenvalues, which change the sign from $+$ to $-$. It can be described as the index of the *suspension* $\{-\frac{\partial}{\partial u}+B_u\}$ which is an elliptic operator over $Y\times S^1$.

It is worth mentioning that for Dirac operators there is a one-one correspondence between the linear conjugation problems and the *cutting and pasting* of Dirac operators; and we obtain the same list of 'correcting' operators and explicit 'error terms' as above. By the decomposition of manifolds and operators into 'elementary' pieces, the cutting and pasting procedure also provides a direct inductive proof of the Atiyah-Singer index theorem for elliptic pseudo-differential operators on closed manifolds.

We close our discussion of the even-dimensional case with a *non-additivity theorem*. Consider two Dirac operators A^j over X_j, $j=1,2$, and assume that A^1 and A^2 are *consistent* with regard to Clifford multiplication, i.e. if A^1 takes the form $G(\partial_u+B)$ close to Y, then A^2 takes the form $G^{-1}(\partial_v-GBG^{-1})$ close to Y, where u denotes the inward normal on X_1 and v the inward normal on X_2. Then a Dirac operator $A^1\cup A^2$ is well defined over M and we obtain as a corollary to Theorem 8:

Theorem 11 *Let E_λ denote the eigenspace of the boundary Dirac operator B_0 to the eigenvalue λ, and $\mathrm{P}_{\geq\lambda}$ the spectral projection onto the direct sum of all E_α with $\alpha\geq\lambda$. Then*

$$\mathrm{index}\, A^1\cup A^2=\mathrm{index}(A^1)_{\mathrm{P}^1_{\geq\lambda}}+\mathrm{index}(A^2)_{\mathrm{P}^2_{\geq-\lambda}}+\dim E_\lambda.$$

Note. For $\lambda=0$ we obtain

$$\mathrm{index}(A^1\cup A^2)=\mathrm{index}(A^1)_{\mathrm{P}_\geq}+\mathrm{index}(A^2)_{G^*(\mathrm{Id}-\mathrm{P}_\geq)G}+\dim\ker B,$$

which corresponds exactly to the Novikov additivity of the signature since $\mathrm{sign}\, X=\mathrm{index}\, A^+_{\mathrm{P}_\geq}+\frac{1}{2}\dim\ker B$.

5. Odd Dimension

If n is odd, the total Dirac operator takes the form $A=G(\partial_u+B)$ near Y. Since G is a unitary bundle automorphism with $G^2=-\mathrm{Id}_{S|_Y}$, it defines a decomposition of $S|_Y$ into the direct sum $S^+\oplus S^-$ of the subbundles of the $\pm i$-eigenvalues of $\{G_y\}_{y\in Y}$. With respect to this decomposition the operator A takes the following form near Y:

$$A=\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix}\left(\partial_u+\begin{pmatrix} 0 & B^-=(B^+)^*\\ B^+ & 0\end{pmatrix}\right). \tag{13}$$

Notice that the formal self-adjointness of $A=G(\partial_u+B)$ implies $GB=-BG$.

The product form (13) has various far reaching consequences. We begin with the classical *cobordism theorem*:

Theorem 12 (Atiyah, Singer [4]) *The index of a Dirac operator*

$$B^+ : C^\infty(Y; S^+) \longrightarrow C^\infty(Y; S^-)$$

over a closed even-dimensional manifold Y vanishes, if the couple (Y, S^+) is a 'boundary', i.e. if there exists a manifold X with boundary Y and a bundle of Clifford modules over X which, restricted to Y, is equal to $S^+ \oplus S^-$.

Proof From (10) we get for any A of the form (13) close to the boundary Y of an odd-dimensional manifold X index A_{Π_-} − index A_{Π_+} = index B^+, where $\Pi_\pm$ denotes the local elliptic boundary conditions introduced in Example 5. But index $A_{\Pi_\pm}$ vanishes by Green's formula (5). □

It follows that $\dim\ker B = 2\dim\ker B^+$. Moreover, the spectrum of B is symmetric with respect to 0; hence $\eta_B(z) \equiv 0$; and the index density α vanishes for odd-dimensional X. This reduces Theorem 8 to the simple formulas

$$\begin{aligned} \text{index}\, A_{P_\geq(B)} &= -\dim\ker B_+ = \mathrm{i}(P_\geq(B), \mathcal{P}_+(A)) \qquad \text{and} \\ \text{index}\, A_R &= -\dim\ker B_+ + \mathrm{i}(R, P_\geq(B)). \end{aligned}$$

That a kernel dimension appears indicates the non-homotopy invariance, see the example in Hitchin [21].

Now we discuss the true *odd* situation, i.e. we are given an odd-dimensional closed Riemannian manifold M which is partitioned in two manifolds X_1, X_2 with boundary by a hypersurface Y, and a bundle S of Clifford modules over M. We fix a bicollar neighbourhood N of Y in M. We assume that the metrics of M and S are product near Y.

The invariants we study on odd-dimensional manifolds are *spectral invariants*: the spectral flow, the η-invariant, and the analytic torsion. They are defined by the spectra of self-adjoint Dirac operators over M or Y; by the spectra of self-adjoint boundary problems; by the spectra of associated Laplacians; or by the spectra of naturally associated families. We shall begin with the spectral flow introduced above after Theorem 12. It plays a prominent role in the recent work of 'three-dimensional' topologists where it is used to determine the value of *Casson's invariant* and other invariants introduced recently (see Taubes [37], Witten [38]).

Let $\mathcal{D}$ denote the space of *cylindrical* Dirac operators acting on S with fixed principal symbol $G(x,\xi)$. Cylindrical means that the operator has the form (13) in the fixed cylinder N; i.e. elements of $\mathcal{D}$ differ over N only by endomorphisms of the bundle S. Let $\mathcal{D}^\times$ denote the subspace of invertible Dirac operators. We consider families

$$\mathcal{A} = \{A_r\} : (I; \{0,1\}) \longrightarrow (\mathcal{D}; \mathcal{D}^\times).$$

It is not difficult to see that $\mathrm{sf}\{A_r\}$ is an integer-valued homotopy invariant of such families over M. In order to obtain an *odd* variant of Bojarski's theorem we need an invariant of families defined over Y. This was worked out recently by L. Nicolaescu [31]. Once again, the key is the 'twisted' orthogonality of Cauchy data of (8) which in this case says

$$\mathcal{P}(A) = -G(\mathrm{Id} - \mathcal{P}(A^*))G = -G(\mathrm{Id} - \mathcal{P}(A))G.$$

This equation has the following nice interpretation. Recall that $G|_Y$ defines a symplectic structure on $L^2(Y;S|_Y)$

$$\{s_1, s_2\} := (Gs_1, s_2) = \int_Y \langle Gs_1(y); s_2(y)\rangle dy.$$

Theorem 13 ([9]) *For $A \in \mathcal{D}$, the Cauchy data space $H(A^1)$ is a Lagrangian subspace of $L^2(Y;S|_Y)$ with respect to G.*

Here $H(A^1)$ denotes the Cauchy data space of the operator $A^1 := A|_{X_1}$. *Lagrangian* means that $G(H(A^1))$ and $H(A^1)$ are orthogonal to each other and span the whole space. It is an easy consequence of Kuiper's theorem that the space of all Lagrangian subspaces is contractible (in the operator topology of the corresponding projections). To get a situation which is interesting from a topological point of view we consider the space $\mathcal{F}_2^G$ of all Fredholm pairs of Lagrangian subspaces of $L^2(Y;S|_Y)$. We have the following result proved by Nicolaescu [31] (for the definition of the K-group KR^{-7} see Karoubi [22]):

Theorem 14 (Nicolaescu [31]) *The space $\mathcal{F}_2^G$ has the homotopy type of a classifying space for KR^{-7}; in particular*

$$\pi_1(\mathcal{F}_2^G) \simeq \mathbf{Z}.$$

It follows that to any loop $\{H_r^1, H_r^2\}$ of elements of $\mathcal{F}_2^G$ corresponds an integer $\mu\{H_r^1, H_r^2\}$, which is an obvious generalization of the standard *Maslov index* defined in finite dimensions (see Arnold [2]). Following standard procedures we can define the Maslov index for any path in $\mathcal{F}_2^G$ and formulate the 'odd' variant of Bojarski's theorem.

Theorem 15 (Nicolaescu [31]) *Let $\{A_r\} : (I;\{0,1\}) \to (\mathcal{D};\mathcal{D}^\times)$ be a smooth family of cylindrical Dirac operators on M acting on S. Then*

$$\mathrm{sf}\{A_r\} = \mu\{H(A_r^1), H(A_r^2)\}.$$

This theorem leads to a decomposition formula for the spectral flow. Let $\mathcal{L}_1$, $\mathcal{L}_2$ denote subspaces of $\ker B$ such that $\mathrm{range}(\mathrm{P}_>) + \mathcal{L}_j$ are Lagrangian subspaces of $L^2(Y;S|_Y)$ (or, equivalently, $\mathcal{L}_j$ is a Lagrangian subspace of $\ker B$ which is a finite-dimensional symplectic space). We denote by $A^j{}_{\mathcal{L}_j}$, $j = 1, 2$ the operator

$$\begin{cases} A^j{}_{\mathcal{L}_j} = A^j := A|_{X_j} \\ \mathrm{dom} A^j{}_{\mathcal{L}_j} := \{s \in \mathcal{H}^1(X_j;S|_{X_j}) \mid (\mathrm{P}_> + \pi_j)(s|_Y) = 0\}, \end{cases}$$

where π_j denotes the projection onto $\mathcal{L}_j$. Then $A^j{}_{\mathcal{L}_j}$ is a closed self-adjoint operator in $L^2(X_j;S|_{X_j})$. Varying A (and so also its 'tangential' Dirac operator B) and choosing families $\{\mathcal{L}_{j,r}\}$, the following result follows from the work of Nicolaescu.

Theorem 16 $\mathrm{sf}\{A_r\} = \mathrm{sf}\{(A_r^1)_{\mathcal{L}_{1,r}}\} + \mathrm{sf}\{(A_r^2)_{\mathcal{L}_{2,r}}\} + \mu\{\mathcal{L}_{1,r}, \mathcal{L}_{2,r}\}$.

This theorem has already been used in the work of Kirk and Klassen [23] in their computation of topological invariants of 3-manifolds.

We have a similar result for the η-invariant. The operator $A^1_{\mathcal{L}}$ has a discrete spectrum and the η-invariant of such operators is well defined (see [15]). The next result follows from the work of Lesch and Wojciechowski [27] and an observation made by W. Müller [30] that, modulo the integers, the η-invariant on manifolds of the form $M = X_1 \cup [-R, R] \times Y \cup X_2$ with cylindrical A does not depend on the length R of the cylinder.

Theorem 17 *Let $\eta(\mathcal{L}_1, \mathcal{L}_2)$ denote the η-invariant of A on the cylinder $[-1,1] \times Y$ with boundary condition $\mathcal{L}_1$ at $u = -1$ and $\mathcal{L}_2$ at $u = 1$. Then we have*

$$\eta(A) \equiv \eta((A^1)_{\mathcal{L}_1}) + \eta((A^2)_{\mathcal{L}_2}) + \eta(\mathcal{L}_1, \mathcal{L}_2) \mod \mathbf{Z},$$

Note. The integer contribution has been computed recently by U. Bunke [13].

6. History and Perspectives

For learning the basic relations between Dirac operators and global analysis on manifolds without boundary we refer to Palais [32], Karoubi [22], Gilkey [16], [17], Lawson and Michelsohn [26], and Berline, Getzler, and Vergne [5]; for details of the calculus on manifolds with boundary as sketched in the sections 1-4 of this article see Booß and Wojciechowski [11]; for other approaches than presented here in Section 5 to the 'odd' problem and to the cutting and pasting of η-invariants and analytic torsion over partitioned manifolds see Gilkey and Smith [19], [20], Roe [33], Singer [36], Booß and Wojciechowski [10], Gilkey [18], Mazzeo and Melrose [29], Klimek and Wojciechowski [24], [25], and Lück [28].

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